# APPROXIMATIONS

(For Board)

## Approximation by Differentials :

A method for approximating the value of a function near a known value.

The method uses the tangent line at the known value of the function to approximate the function's graph.

In this method, $\Delta x$ and $\Delta y$ represent the changes in $x$ and $y$ for the function, and $dx$ and $dy$ represent the changes in $x$ and $y$ for the tangent line.

If $y = f(x)$ and $\Delta x$ is a small change in $x$ then the corresponding change in $y$ (approximately) is given by $f'(x)\Delta x$.

This is called the differential of $y$ and is denoted by $dy$

$$\therefore\ dy = f'(x)\,\Delta x$$

The actual change or the actual error in $y$ is denoted by

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\Delta y \cong dy = f'(x)\,\Delta x$$

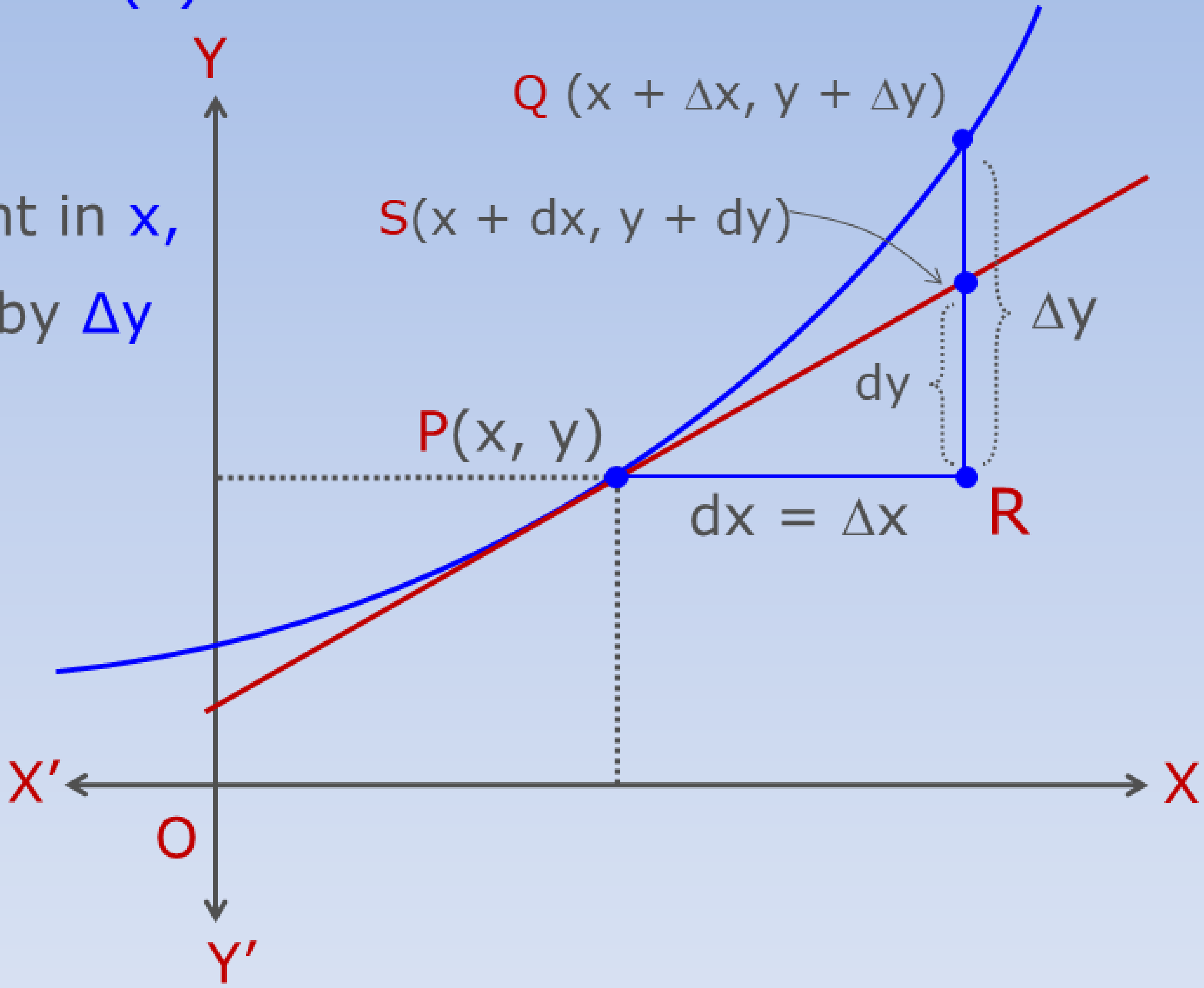
## Geometrical Meaning of approximations

Let $f : D \to R$, $D \subset R$, be a given function and let $y = f(x)$.

Let $\Delta x$ denote a small increment in $x$.

The increment in $y$ corresponding to the increment in $x$, denoted by $\Delta y$

$$\Delta y = f(x + \Delta x) - f(x)$$




(i) The differential of $x$, denoted by $dx$, is defined by $dx = \Delta x$.

(ii) The differential of $y$, denoted by $dy$, is defined by $dy = f'(x)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$

In case $dx = \Delta x$ is relatively small when compared with $x$, $dy$ is a good approximation of $\Delta y$ and we denote it by $dy \approx \Delta y$.

Use differential to approximate $\sqrt{36.6}$.

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$y = \sqrt{x}$ Let $x = 36$ & $\Delta x = 0.6$

$\Delta y = f(x + \Delta x) - f(x)$

$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{36.6} - \sqrt{36}$

$= \sqrt{36.6} - 6$

$\Rightarrow \sqrt{36.6} = 6 + \Delta y$

$dy \approx \Delta y$ and is given by $dy = \left(\frac{dy}{dx}\right)\Delta x$

$dy = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2 \times 6}(0.6) = 0.05$

$y = \sqrt{x} \Rightarrow y = x^{1/2}$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{1/2 - 1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

Thus, the approximate value of $\sqrt{36.6} = 6 + \Delta y = 6 + 0.05 = 6.05.$

Use differential to approximate $(25)^{1/3}$.

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$y = x^{1/3}$ Let $x = 27$ & $\Delta x = -2$

$\Delta y = f(x + \Delta x) - f(x)$

$\Delta y = (x + \Delta x)^{1/3} - x^{1/3} = (25)^{1/3} - (27)^{1/3}$

$= (25)^{1/3} - 3$

$\Rightarrow (25)^{1/3} = 3 + \Delta y$

$dy \approx \Delta y$ and is given by $dy = \left(\frac{dy}{dx}\right)\Delta x$

$dy = \frac{1}{3\,x^{2/3}}(-2) = \frac{1}{3 \times 27^{2/3}}(-2) = \frac{-2}{27} = -0.074$

$y = x^{1/3} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\, x^{1/3}$

$= \frac{1}{3}\, x^{1/3 - 1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3} x^{-2/3} = \frac{1}{3\,x^{2/3}}$

Thus, the approximate value of $(25)^{1/3} = 3 + \Delta y = 3 - 0.074 = 2.926.$

Find the approximate value of f(3.02), where $f(x) = 3x^2 + 5x + 3$.

Find the approximate value of $f(3.02)$, where $f(x) = 3x^2 + 5x + 3$.

Let $x = 3$ and $\Delta x = 0.02$.

$$y = 3x^2 + 5x + 3$$

$$\frac{dy}{dx} = 3\frac{d}{dx}x^2 + 5\frac{d}{dx}x + \frac{d}{dx}3 = 3(2x) + 5(1) + 0 = 6x + 5$$

$dy \approx \Delta y$ and is given by $dy = \left(\frac{dy}{dx}\right)\Delta x$

$$\Delta y = f(x + \Delta x) - f(x)$$

$$\Rightarrow f(x + \Delta x) = \Delta y + f(x)$$

$$\Rightarrow f(x + \Delta x) = \left(\frac{dy}{dx}\right)\Delta x + f(x)$$

$$f(x + \Delta x) = (6x + 5)(0.02) + 3x^2 + 5x + 3$$

$$f(3 + 0.02) = (6 \times 3 + 5)(0.02) + 3(3)^2 + 5 \times 3 + 3$$

$$\Rightarrow f(3.02) = (23)(0.02) + 3(9) + 15 + 3$$

$$= 45.46$$

Find the approximate change in the volume V of a cube of side x meters caused by increasing the side by 2%.

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Let side of the cube = x metres

Increase in size = 2% = 0.02x

$\Delta x = 0.02x$

Volume of a cube $V = x^3$

$$\frac{dV}{dx} = \frac{d}{dx}x^3 = 3x^2$$

Approximate change in volume is $\Delta V$

$$\Delta V = \left(\frac{dV}{dx}\right)\Delta x = (3x^2)\,\Delta x$$

$$= (3x^2)\,(0.02x) = 0.06x^3 \text{ m}^3$$

If the radius of a sphere is measured as 9 cm with an error of 0.03cm, then find the approximate error in calculating its volume.

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Let r be the radius of the sphere and

$\Delta r$ be the error in measuring the radius.

$r = 9$ cm and $\Delta r = 0.03$ cm.

The volume of the sphere is $V = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi \frac{d}{dr} r^3 = \frac{4}{3}\pi\, 3r^2$$

$$\Rightarrow \frac{dV}{dr} = 4\pi r^2$$

The approximate error in the volume is $\Delta V$

$$\therefore \quad \Delta V = \left(\frac{dV}{dr}\right)\Delta r = (4\pi r^2)\,\Delta r$$

$$= 4\pi(9)^2\,(0.03) = 9.72\pi \text{ cm}^3$$